A NOTE ON THE FITTING OF PARABOLAS

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The formulae given by Pearson[2] (pp. 12–16) and Elderton[3] (pp. 30–31) for the fitting of parabolas by the method of moments assume the origin at the mid-point of the range. It being often more convenient to take the origin one unit below the first ordinate, as in working by the method of least squares, I have, at the suggestion of Dr. Raymond Pearl, worked out the formulae which result from such choice of origin.

Let l be the range for which the parabola

$$y = c_0 + c_1x + c_2x^2 + \dots + c_nx^n$$

is to be fitted to the observations, and $M_r = S(yx^r)$ where the summation includes the values of x and y for every observation.

Then

$$M_r = \int_{\frac{1}{2}}^{l+\frac{1}{2}} (c_0 + c_1x + c_2x^2 + \dots c_nx^n)\, x^r dx$$

$$= \frac{c_0}{r+1}[(l+\tfrac{1}{2})^{r+1} - (\tfrac{1}{2})^{r+1}] + \frac{c_1}{r+2}[(l+\tfrac{1}{2})^{r+2} - (\tfrac{1}{2})^{r+2}] + \dots$$

$$+ \frac{c_n}{r+n+1}[(l+\tfrac{1}{2})^{r+n+1} - (\tfrac{1}{2})^{r+n+1}]$$

Substituting $r = 0, 1, 2, \dots n$ in this formula we have $n+1$ simultaneous equations from the solution of which we may express the c's in terms of the moments and certain functions of l.

(i). $y = c_0 + c_1x,$
$c_0 = K_1M_0 - K_2M_1,$
$c_1 = -K_2M_0 + K_3M_1,$

where

$$K_1 = \frac{1}{l^3}(4l^2 + 6l + 3), \qquad K_2 = \frac{6}{l^3}(l+1), \qquad K_3 = 12/l^3.$$

(ii). $y = c_0 + c_1x + c_2x^2,$
$c_0 = K_4M_0 - K_5M_1 + K_6M_2,$
$c_1 = -K_5M_0 + K_7M_1 - K_8M_2,$
$c_2 = K_6M_0 - K_8M_1 + K_9M_2,$

where

$$K_4 = \frac{9}{l^5}\left(l^4 + 4l^3 + 7l^2 + 5l + \frac{5}{4}\right), \qquad K_5 = \frac{9}{l^5}(l+1)(4l^2 + 10l + 5),$$

$$K_6 = \frac{30}{l^5}\left(l^2 + 3l + \frac{3}{2}\right), \qquad K_7 = \frac{12}{l^5}(16l^2 + 30l + 15),$$

$$K_8 = \frac{180}{l^5}(l+1), \qquad K_9 = 180/l^5.$$

(iii). $y = c_0 + c_1x + c_2x^2 + c_3x^3,$

$$c_0 = K_{10}M_0 - K_{11}M_1 + K_{12}M_2 - K_{13}M_3,$$
$$c_1 = -K_{11}M_0 + K_{14}M_1 - K_{15}M_2 + K_{16}M_3,$$
$$c_2 = K_{12}M_0 - K_{15}M_1 + K_{17}M_2 - K_{18}M_3,$$
$$c_3 = -K_{13}M_0 + K_{16}M_1 - K_{18}M_2 + K_{19}M_3,$$

where

$$K_{10} = \frac{1}{4l^7}(64l^6 + 480l^5 + 1680l^4 + 2840l^3 + 2460l^2 + 1050l + 175),$$

$$K_{11} = \frac{15}{2l^7}(l+1)(16l^4 + 96l^3 + 188l^2 + 140l + 35),$$

$$K_{12} = \frac{15}{l^7}(16l^4 + 104l^3 + 192l^2 + 140l + 35),$$

$$K_{13} = \frac{70}{l^7}(l+1)(2l^2 + 10l + 5),$$

$$K_{14} = \frac{75}{l^7}(16l^4 + 72l^3 + 120l^2 + 84l + 21),$$

$$K_{15} = \frac{900}{l^7}(l+1)\left(3l^2 + 7l + \frac{7}{2}\right),$$

$$K_{16} = \frac{420}{l^7}(4l^2 + 10l + 5), \qquad K_{17} = \frac{180}{l^7}(36l^2 + 70l + 35),$$

$$K_{18} = \frac{4200}{l^7}(l+1), \qquad K_{19} = 2800/l^7.$$

The values of the K's for values of l up to 30 are given in Table II.

The fitting of the following observations, given by Thiele[4] (p. 12) and used by Pearson[2] to illustrate his formulae for fitting parabolas, will serve as an example. Table I shows the calculations to obtain the moments and the resulting parabolas. It is obvious that these data are in no way suited to graduation by parabolas, being really a unimodal frequency distribution. They will, however, serve for illustration of method.

The origin for moments is taken at $X = 6$ and the successive moments corrected by Sheppard's formula (λ)[5] (p. 276).

TABLE I

X	y	x	yx	yx^2	yx^3	Parabolas		
						1st	2nd	3rd
7	3	1	3	3	3	57.5	11.0	−14.1
8	7	2	14	28	56	54.3	30.9	32.2
9	35	3	105	315	945	51.2	46.6	62.2
10	101	4	404	1,616	6,464	48.0	58.1	78.4
11	89	5	445	2,225	11,125	44.9	65.4	83.1
12	94	6	564	3,384	20,304	41.7	68.5	78.9
13	70	7	490	3,430	24,010	38.5	67.5	68.0
14	46	8	368	2,944	23,552	35.4	62.2	53.0
15	30	9	270	2,430	21,870	32.2	52.8	36.2
16	15	10	150	1,500	15,000	29.1	39.1	20.0
17	4	11	44	484	5,324	25.9	21.3	6.8
18	5	12	60	720	8,640	22.8	−0.7	−0.8
19	1	13	13	169	2,197	19.6	−26.9	−0.7
Totals......	500	......	2,930	19,248	139,490	....	....	....

$$M_0 = 501.031015 \qquad M_2 = 19190.3584$$

$$l = 13$$

$$M_1 = 2929.06288 \qquad M_3 = 138630.787$$

(i).

$$c_0 = 0.344561M_0 - 0.0382340M_1 = 60.6459$$
$$c_1 = -0.0382340M_0 + 0.00546199M_1 = -3.15791$$

(ii).

$$c_0 = 0.935607M_0 - 0.275217M_1 + 0.0169273M_2 = -13.1046$$
$$c_1 = -0.275217M_0 + 0.100481M_1 - 0.00678709M_2 = 26.1762$$
$$c_2 = 0.0169273M_0 - 0.00678709M_1 + 0.000484792M_2 = -2.095379$$

(iii).

$$c_0 = 2.158569M_0 - 1.173878M_1 + 0.172060M_2 - 0.00738727M_3 = -79.062$$
$$c_1 = -1.173878M_0 + 0.760838M_1 - 0.120782M_2 + 0.00542834M_3 = 75.0781$$
$$c_2 = 0.172060M_0 - 0.120782M_1 + 0.0201633M_2 - 0.000937074M_3 = -10.53703$$
$$c_3 = -0.00738727M_0 + 0.00542834M_1 - 0.000937074M_2 + 0.0000446226M_3 = 0.401978$$

[1] Papers from the Biological Laboratory of the Maine Agricultural Experiment Station No. 106.

[2] Pearson, K., *Biometrika, Cambridge*, 2, 1902, (1–23).

[3] Elderton, W. P., *Frequency Curves and Correlation*, London, 1907, pp. 172.

[4] Thiele, T. N., *Forelaesninger over Almindelig Iagttagelslaere*, Kjφbenhavn, 1889.

[5] Pearson, K., *Biometrika, Cambridge*, 1, 1902, (265–303).

TABLE II

VALUES OF THE K'S FOR VALUES OF l FROM 2 TO 30

l	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}
2	3.875000	2.250000	1.500000							
3	2.111111	0.888889	0.444444	9.935185	10.518519	2.407407	12.296296	2.962963	0.740741	
4	1.421875	0.468750	0.187500	5.671143	4.790039	0.864258	4.582031	0.878906	0.175781	21.503281
5	1.064000	0.288000	0.0960000	3.819600	2.678400	0.398400	2.169600	0.345600	0.0576000	12.710160
6	0.847222	0.194444	0.0555556	2.827836	1.693287	0.214120	1.189815	0.162037	0.0231481	8.577156
7	0.702624	0.139942	0.0349854	2.223493	1.160945	0.127625	0.720414	0.0856786	0.0107098	6.293770
8	0.599609	0.105469	0.0234375	1.821876	0.842926	0.0819397	0.468384	0.0494385	0.00549316	4.888443
9	0.522634	0.0823045	0.0164609	1.537913	0.638622	0.0556318	0.321292	0.0304832	0.00304832	3.954524
10	0.463000	0.0660000	0.0120000	1.327613	0.499950	0.0394500	0.229800	0.0198000	0.00180000	3.297417
11	0.415477	0.0540947	0.00901578	1.166179	0.401686	0.0289660	0.169959	0.0134119	0.00111766	2.814296
12	0.376736	0.0451389	0.00694444	1.038674	0.329608	0.0218822	0.129196	0.00940394	0.000723380	2.446523
13	0.344561	0.0382340	0.00546199	0.935607	0.275217	0.0169273	0.100481	0.00678709	0.000484792	2.158569
14	0.317420	0.0327988	0.00437318	0.850682	0.233190	0.0133594	0.0796766	0.00502023	0.000334682	1.927824
15	0.294222	0.0284444	0.00355556	0.779570	0.200059	0.0107259	0.0642370	0.00379259	0.000237037	1.739302
16	0.274170	0.0249023	0.00292969	0.719203	0.173490	0.00874043	0.0525398	0.00291824	0.000171661	1.582725
17	0.256667	0.0219825	0.00244250	0.667349	0.151862	0.00721552	0.0435171	0.00228192	0.000126773	1.450834
18	0.241255	0.0195473	0.00205761	0.622348	0.134026	0.00602519	0.0364464	0.00180994	0.0000952599	1.338372
19	0.227584	0.0174953	0.00174953	0.582942	0.119147	0.00508259	0.0308275	0.00145390	0.0000726950	1.241451
20	0.215375	0.0157500	0.00150000	0.548160	0.106608	0.00432656	0.0263063	0.00118125	0.0000562500	1.157134
21	0.204406	0.0142533	0.00129576	0.517241	0.0959433	0.00371318	0.0226273	0.000969614	0.0000440733	1.083170
22	0.194497	0.0129602	0.00112697	0.489582	0.0867984	0.00321036	0.0196033	0.000803317	0.0000349268	1.017803
23	0.185502	0.0118353	0.000986274	0.464698	0.0788982	0.00279429	0.0170948	0.000671189	0.0000279662	0.959649
24	0.177300	0.0108507	0.000868056	0.442194	0.0720271	0.00244706	0.0149966	0.000565140	0.0000226056	0.907601
25	0.169792	0.00998400	0.000768000	0.421748	0.0660142	0.00215501	0.0132280	0.000479232	0.0000184320	0.860762
26	0.162893	0.00921711	0.000682749	0.403092	0.0607225	0.00190761	0.0117269	0.000409044	0.0000151498	0.818402
27	0.156531	0.00853528	0.000609663	0.386002	0.0560413	0.00169664	0.0104446	0.000351246	0.0000125445	0.779919
28	0.150647	0.00792638	0.000546647	0.370291	0.0518804	0.00151566	0.00934251	0.000303306	0.0000104588	0.744814
29	0.145188	0.00738038	0.000492025	0.355798	0.0481655	0.00135950	0.00839017	0.000263271	0.00000877572	0.712667
30	0.140111	0.00688889	0.000444444	0.342389	0.0448352	0.00122407	0.00756296	0.000229630	0.00000740741	0.683124

TABLE II—*Continued*

l	K_{11}	K_{12}	K_{13}	K_{14}	K_{15}	K_{16}	K_{17}	K_{18}	K_{19}
4	31.684113	13.200989	1.644897	50.267029	21.835327	2.794189	9.788818	1.281738	0.170898
5	15.802560	5.478720	0.564480	21.543360	7.845120	0.833280	2.960640	0.322560	0.0358400
6	9.211168	2.732071	0.239805	11.020287	3.454540	0.313572	1.125900	0.105024	0.0100023
7	5.942932	1.539282	0.117638	6.338558	1.744171	0.138208	0.500302	0.0407993	0.00339994
8	4.115850	0.945761	0.0639868	3.961551	0.971389	0.0682926	0.248823	0.0180244	0.00133514
9	3.002575	0.619821	0.0376126	2.633735	0.582379	0.0367930	0.134766	0.00878116	0.000585410
10	2.278939	0.426953	0.0234850	1.836457	0.369765	0.0212100	0.0780300	0.00462000	0.000280000
11	1.784351	0.305961	0.0153886	1.329925	0.245792	0.0129100	0.0476714	0.00258632	0.000143684
12	1.432497	0.226412	0.0104887	0.993184	0.169630	0.00821672	0.0304372	0.00152379	0.0000781429
13	1.173878	0.172060	0.00738727	0.760838	0.120782	0.00542834	0.0201633	0.000937074	0.0000446226
14	0.978564	0.133710	0.00534894	0.595470	0.0883023	0.00370142	0.0137817	0.000597646	0.0000265621
15	0.827653	0.105906	0.00396583	0.474636	0.0660333	0.00259336	0.00967638	0.000393306	0.0000163877
16	0.708755	0.0852712	0.00300121	0.384331	0.0503568	0.00186034	0.00695430	0.000265986	0.0000104308
17	0.613490	0.0696446	0.00231219	0.315508	0.0390650	0.00136234	0.00510120	0.000184238	0.00000682363
18	0.536035	0.0575996	0.00180963	0.262152	0.0307661	0.00101601	0.00381010	0.000130345	0.00000457352
19	0.472243	0.0481693	0.00143622	0.220159	0.0245572	0.000770110	0.00289189	0.0000939732	0.00000313244
20	0.419102	0.0406832	0.00115418	0.186662	0.0198376	0.000592266	0.00222680	0.0000689063	0.00000218750
21	0.374382	0.0346664	0.000937977	0.159620	0.0161987	0.000461488	0.00173705	0.0000513023	0.00000155462
22	0.336404	0.0297764	0.000770030	0.137551	0.0133568	0.000363869	0.00137102	0.0000387274	0.00000112253
23	0.303885	0.0257619	0.000637989	0.119365	0.0111114	0.000290006	0.00109375	0.0000296050	0.000000822362
24	0.275833	0.0224359	0.000533035	0.104245	0.00931844	0.000233421	0.000881109	0.0000228934	0.000000610491
25	0.251470	0.0196572	0.000448774	0.0915717	0.00787282	0.000189579	0.000716194	0.0000178913	0.000000458752
26	0.230180	0.0173180	0.000380503	0.0808698	0.00669688	0.000155255	0.000586964	0.0000141189	0.000000348614
27	0.211468	0.0153349	0.000324719	0.0717710	0.00573245	0.000128124	0.000484727	0.0000112425	0.000000267677
28	0.194938	0.0136501	0.000278782	0.0639863	0.00493549	0.000106487	0.000403131	0.00000902695	0.000000207516
29	0.180263	0.0121901	0.000240680	0.0572869	0.00427229	0.0000890893	0.000337474	0.00000730440	0.000000162320
30	0.167177	0.0109362	0.000208864	0.0514902	0.00371681	0.0000749931	0.000284239	0.00000595336	0.000000128029